TSRT14: Sensor Fusion
Lecture 5

— Sensor and motion models

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Lecture 5: sensor and motion models

Whiteboard:

- Principles and some examples

Slides:

- Sampling formulas
- Noise models
- Standard motion models
  - Position as integrated velocity, acceleration, . . . , in nD.
  - Orientation as integrated angular speed in 2D and 3D.
- Odometry

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Lecture 4: summary

- Detection problems as hypothesis tests:
$$H_0 : \mathbf{y} = \mathbf{e},$$
$$H_1 : \mathbf{y} = \bar{x} + \mathbf{e} = \mathbf{h}(x) + \mathbf{e}.$$
- Neyman-Pearson's lemma: $T(y) = p_{\mathbf{e}}\big(y - \mathbf{h}(x^0)\big)/p_{\mathbf{e}}(y)$ maximizes P_D for given P_{FA} (best ROC curve).
- In general case
$$\bar{T}(y) = 2\log p_{\mathbf{e}}\big(y - \mathbf{h}(\hat{x}^{\mathrm{ML}})\big) - 2\log p_{\mathbf{e}}(y) \sim \chi^2_{n_x}\big(x^{0,T}\,\mathcal{I}(x^0)x^0\big).$$
- Bayes optimal filter
$$p(x_k|y_{1:k}) \propto p_{e_k}\big(y_k - h(x_k)\big)p(x_k|y_{1:k-1})$$
$$p(x_{k+1}|y_{1:k}) = \int p_{v_k}\big(x_{k+1} - f(x_k)\big)p(x_k|y_{1:k})\,dx_k.$$
- Intuitive work flow of nonlinear filter:
  - MU: estimation from $y_k = h(x_k) + e_k$ and fusion with $\hat{x}_{k|k-1}$.
  - TU: nonlinear transformation $z = f(x_k)$ and diffusion from $x_{k-1} = z_k + v_k$.

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Modeling and Motion Models

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Chapters 12–14 Overview

- Chapter 12: Principles and methods
  - Principles for deriving discrete time models from continuous time ones
  - Discretized-linearization
  - Linearized-discretization
  - Calibration
- Chapter 13: Motion models
  - Kinematics
  - Rotations
  - Vehicle models
  - Examples
- Chapter 14: Sensor models
  - Techniques
  - Examples

Modeling

First problem:
Physics give continuous time model, filters require (linear or nonlinear) discrete time model:

Classification	Nonlinear	Linear
Continuous time	$\dot{x} = a(x,u) + v$	$\dot{x} = Ax + Bu + v$
	$y = c(x,u) + e$	$y = Cx + Du + e$
Discrete time	$x_{k+1} = f(x,u) + \bar{v}$	$x_{k+1} = Fx + Gu + \bar{v}$
	$y = h(x,u) + e$	$y = Hx + Ju + e$

Sampling Formulas (1/2)

Linear time-invariant (LTI) state-space model:

Continuous time	Discrete time
$\dot{x} = Ax + Bu$	$x_{k+1} = Fx_k + Gu_k$
$y = Cx + Du$	$y_k = Hx_k + Ju_k$

u is either input or process noise (then J denotes cross-correlated noise!).

- **Zero-order hold (ZOH) sampling** assuming the input is piecewise constant:

$$x(t+T) = e^{AT}x(t) + \int_0^T e^{A\tau}Bu(t+T-\tau)\,d\tau$$
$$= \underbrace{e^{AT}}_{F}x(t) + \underbrace{\int_0^T e^{A\tau}\,d\tau\, B}_{G}u(t).$$

- **First order hold (FOH) sampling** assuming the input is piecewise linear, is another option.

Sampling Formulas (2/2)

- **Bilinear transformation (BIL)** assumes band-limited input

$$\frac{2}{T}\frac{\Delta - 1}{\Delta + 1}x(t) \approx \tfrac{d}{dt}x(t) = Ax + Bu,$$

where Δ is the delay operator, $\Delta x(t) = x(t+T)$, which yields

$$\begin{aligned} M &= (I_{n_x} - T/2A)^{-1} \\ F &= M(I_{n_x} + T/2A) \\ G &= T/2MB \\ H &= CM \\ J &= D + HG. \end{aligned}$$

Sampling of Nonlinear Models

There are two options:

- Discretized linearization (general):
  1. Linearize:

$$A = \nabla_x a(x,u) \qquad B = \nabla_u a(x,u) \qquad C = \nabla_x c(x,u) \qquad D = \nabla_u c(x,u)$$

  2. Discretize (sample): $F = e^{AT}$, $G = \int_0^T e^{A\tau}\, d\tau\, B$, $H = C$, and $J = D$
- Linearized discretization (best, if possible!):
  1. Discretize (sample nonlinear):

$$x(t+T) = f\big(x(t), u(t)\big) = x(t) + \int_t^{t+T} a\big(x(\tau), u(\tau)\big)\, d\tau$$

  2. Linearize: $F = \nabla_x f(x_k, u_k)$ and $G = \nabla_u f(x_k, u_k)$

Sampling of State Noise

Different solutions exist, they are all approximations except in the linear case:

- v_t is white noise such that its total influence during one sample interval is TQ (alternative (12.14d) in the book):

$$\bar{Q}_d = TQ$$

- v_t is a discrete white noise sequence with variance TQ. That is, we assume that the noise enters immediately after a sample time, so $x(t+T) = f\big(x(t) + v(t)\big)$ (alternative (12.14e) in the book):

$$\bar{Q}_e = TGQG^T$$

Recommendation

In practice simple solutions works well, but *remember to scale with* T!

Motion Models

Continuous time (physical) and discrete time counterparts

$$\dot{x}(t) = a(x(t), u(t), w(t); \theta)$$
$$x(t+T) = f(x(t), u(t), w(t); \theta, T).$$

- **Kinematic models**: Do not attempt to model forces, but are 'Black-box' multi-purpose models.
  - *Translation kinematics* describes position, often based on $F = ma$.
  - *Rotational kinematics* describes orientation.
  - *Rigid body kinematics* combines translational and rotational kinematics.
  - *Constrained kinematics.* Coordinated turns (circular path motion).
- **Application specific force models**
- **Gray-box models** Parameters θ must be calibrated (estimated, identified) from data.

Translational Motion with n Integrators

Translational kinematics models in nD, where $p(t)$ denotes:

- Position: $X(t)$, $\big(X(t), Y(t)\big)^T$, or $\big(X(t), Y(t), Z(t)\big)^T$
- Rotation: $\psi(t)$ or $\big(\phi(t), \theta(t), \psi(t)\big)^T$

The signal $w(t)$ is process noise for a pure kinematic model and a motion input signal in position, velocity, and acceleration, respectively, for the case of using sensed motion as an input rather than as a measurement.

State, x	**Continuous time, $\dot{\mathbf{x}}$**	**Discrete time, $\mathbf{x(t+T)}$**
p	w	$x + Tw$
$\begin{pmatrix} p \\ v \end{pmatrix}$	$\begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix} x + \begin{pmatrix} 0_n \\ I_n \end{pmatrix} w$	$\begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix} x + \begin{pmatrix} \frac{T^2}{2} I_n \\ TI_n \end{pmatrix} w$
$\begin{pmatrix} p \\ v \\ a \end{pmatrix}$	$\begin{pmatrix} 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \\ 0_n & 0_n & 0_n \end{pmatrix} x + \begin{pmatrix} 0_n \\ 0_n \\ I_n \end{pmatrix} w$	$\begin{pmatrix} I_n & TI_n & \frac{T^2}{2} I_n \\ 0_n & I_n & TI_n \\ 0_n & 0_n & I_n \end{pmatrix} x + \begin{pmatrix} \frac{T^3}{6} I_n \\ \frac{T^2}{2} I_n \\ TI_n \end{pmatrix} w$

Different Sampled Models of Double Integrator

Models

$$\dot{x} = Ax + Bu \qquad x_{k+1} = Fx_k + Gu_k$$
$$y = Cx + Du \qquad y_k = Hx_k + Ju_k$$

State: $x = \begin{pmatrix} p(t) \\ v(t) \end{pmatrix}$

Continuous time	$A = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix}$	$B = \begin{pmatrix} 0_n \\ I_n \end{pmatrix}$	$C = (I_n,\ 0_n)$	$D = 0_n$
ZOH	$F = \begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix}$	$G = \begin{pmatrix} \frac{T^2}{2} I_n \\ TI_n \end{pmatrix}$	$H = (I_n,\ 0_n)$	$J = 0_n$
FOH	$F = \begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix}$	$G = \begin{pmatrix} T^2 I_n \\ TI_n \end{pmatrix}$	$H = (I_n,\ 0_n)$	$J = \frac{T^2}{6} I_n$
BIL	$F = \begin{pmatrix} I_n & TI_n \\ 0_n & I_n \end{pmatrix}$	$G = \begin{pmatrix} \frac{T^2}{4} I_n \\ \frac{T}{\cdot} I \end{pmatrix}$	$H = \left(I_n,\ \frac{T}{2} I_n\right)$	$J = \frac{T^2}{2} I_n$

Navigation Models

- Navigation models have access to inertial information.
- 2D orientation (course, or yaw rate) much easier than 3D orientation.

Rotational Kinematics in 2D

The course, or yaw, in 2D can be modeled as integrated white noise

$$\dot{\psi}(t) = w(t),$$

or any higher order of integration. Compare to the tables for translational kinematics with $p(t) = \psi$ and $n = 1$.

Coordinated Turns in 2D Body Coordinates

Basic motion equations

$$\dot{\psi} = \frac{v_x}{R} = v_x R^{-1},$$
$$a_y = \frac{v_x^2}{R} = v_x^2 R^{-1} = v_x \dot{\psi},$$
$$a_x = \dot{v}_x - v_y \frac{v_x}{R} = \dot{v}_x - v_y v_x R^{-1} = \dot{v}_x - v_y \dot{\psi}.$$

can be combined to a model suitable for the sensor configuration at hand. For instance,

$$x = \begin{pmatrix} \psi \\ R^{-1} \end{pmatrix}, \quad u = v_x, \quad y = R^{-1}$$
$$\dot{x} = f(x, u) + w = \begin{pmatrix} v_x R^{-1} \\ 0 \end{pmatrix} + w$$

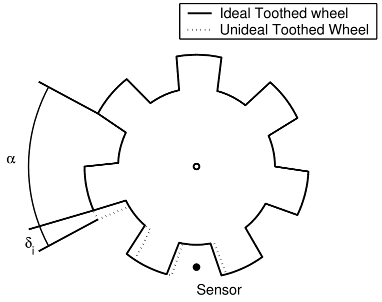
is useful when speed is measured, and a vision system delivers a local estimate of (inverse) curve radius.

Automotive Example: wheel speed sensor

Each tooth passing the sensor (electromagnetic or Hall) gives a pulse. The number n of clock cycles between a number m of teeth are registered within each sample interval.

$$\omega(t_k) = \frac{2\pi}{N_{\mathsf{cog}}(t_k - t_{k-1})} = \frac{2\pi}{N_{\mathsf{cog}} T_c} \frac{m}{n}.$$

Problems:
Angle and time quantization. Synchronization. Angle offsets δ in sensor teeth.



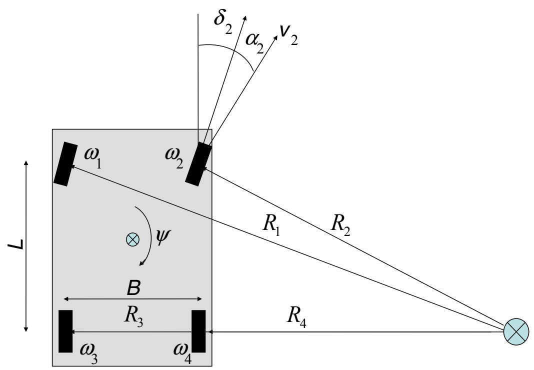
Automotive Example: virtual sensors

Longitudinal velocity, yaw rate and slip on left and right driven wheel (front wheel driven assumed) can be computed from wheel angular speeds **if** the radii are known:

$$\begin{aligned}
v_x &= \frac{v_3 + v_4}{2} = \frac{\omega_3 r_3 + \omega_4 r_4}{2}, \\
\dot{\Psi} &= \frac{\omega_3 r_3 - \omega_4 r_4}{B}, \\
s_1 &= \frac{\omega_1 r_1}{v_1} - 1, \quad s_2 = \frac{\omega_2 r_2}{v_2} - 1, \\
v_1 &= v_x \sqrt{(1 + \tfrac{1}{2} R^{-1} B)^2 + (R^{-1} L)^2}, \\
v_2 &= v_x \sqrt{(1 - \tfrac{1}{2} R^{-1} B)^2 + (R^{-1} L)^2}.
\end{aligned}$$

Automotive Example: geometry

The formulas are based on geometry, the relation $\dot{\psi} = v_x R^{-1}$ and notation below.



Automotive Example: odometry

Odometry is based on the virtual sensors

$$\begin{aligned}
v_x^m &= \frac{\omega_3 r_3 + \omega_4 r_4}{2} \\
\dot{\psi}^m &= v_x^m \frac{2}{B} \frac{\frac{\omega_3}{\omega_4}\frac{r_3}{r_4} - 1}{\frac{\omega_3}{\omega_4}\frac{r_3}{r_4} + 1}.
\end{aligned}$$

and the model

$$\begin{aligned}
\psi_t &= \psi_0 + \int_0^t \dot{\psi}_\tau \, d\tau, \\
X_t &= X_0 + \int_0^t v_\tau^x \cos(\psi_\tau) \, d\tau, \\
Y_t &= Y_0 + \int_0^t v_\tau^x \sin(\psi_\tau) \, d\tau.
\end{aligned}$$

to dead-reckon the wheel speeds to a relative position in the global frame.
The position $(X_t(r_3, r_4), Y_t(r_3, r_4))$ depends on the values of wheel radii r_3 and r_4.
Further sources of error come from wheel slip in longitudinal and lateral direction. More sensors needed for navigation.

Rotational Kinematics in 3D

Much more complicated in 3D than 2D! Could be a course in itself.
Coordinate notation for rotations of a body in local coordinate system (x, y, z) relative to an earth fixed coordinate system:

Motion components	Rotation Euler angle	Angular speed
Longitudinal forward motion x	Roll ϕ	ω^x
Lateral motion y	Pitch θ	ω^y
Vertical motion z	Yaw ψ	ω^z

Euler Orientation in 3D

An earth fixed vector $\mathbf{g}$ (for instance the gravitational force) is in the body system oriented as $Q\mathbf{g}$, where

$$Q = Q^x_\phi Q^y_\theta Q^z_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{pmatrix}.$$

Note:

The result depends on the order of rotations $Q^x_\phi Q^y_\theta Q^z_\psi$. Here, the xyz rule is used, but there are other options.

Euler Rotation in 3D

When the body rotate with ω, the Euler angles change according to

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + Q^x_\phi \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + Q^x_\phi Q^y_\theta \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}.$$

The dynamic equation for Euler angles can be derived from this as

$$\begin{pmatrix} \dot\phi \\ \dot\psi \\ \dot\theta \end{pmatrix} = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.$$

Singularities (gimbal lock) when $\theta = \pm\frac{\pi}{2}$, can cause numeric divergence!

Unit Quaternions

- Vector representation: $q = (q^0, q^1, q^2, q^3)^T$.
- Norm constraint of unit quaternion: $\|q\| = q^T q = 1$.
- The quaternion can be interpreted as as an axis angle:

$$q = \begin{pmatrix} \cos(\frac{1}{2}\alpha) \\ \sin(\frac{1}{2}\alpha)\hat{v} \end{pmatrix},$$

where q represents a rotation with α around the axis defined by $\hat{v}$, $\|\hat{v}\| = 1$.

Pros and Cons

+ No singularity.
+ No 2π ambiguity.
– More complex and non-intuitive algebra.
– The norm must be maintained; this can be handled. by projection or as a virtual measurement with small noise.

Quaternion Orientation in 3D

The orientation of the vector $\mathbf{g}$ in body system is $Q\mathbf{g}$, where

$$Q = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_0q_2+2q_1q_3 \\ 2q_0q_3+2q_1q_2 & q_0^2-q_1^2+q_2^2-q_3^2 & -2q_0q_1+2q_2q_3 \\ -2q_0q_2+2q_1q_3 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix}$$
$$= \begin{pmatrix} 2q_0^2+2q_1^2-1 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 2q_0^2+2q_2^2-1 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 2q_0^2+2q_3^2-1 \end{pmatrix}.$$

Quaternion Rotation in 3D

Rotation with ω gives a dynamic equation for q which can be written in two equivalent forms:

$$\dot{q} = \frac{1}{2}S(\omega)q = \frac{1}{2}\bar{S}(q)\omega,$$

where

$$S(\omega) = \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix}, \qquad \bar{S}(q) = \begin{pmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{pmatrix}.$$

Sampled Form of Quaternion Dynamics

The ZOH sampling formula

$$q(t+T) = e^{\frac{1}{2}S\big(\omega(t)\big)T}q(t)$$

actually has a closed form solution

$$q(t+T) = \Big(\cos(\tfrac{T}{2}\|\omega(t)\|)I_4 + \tfrac{T}{2}\overbrace{\frac{\sin(\frac{T}{2}\|\omega(t)\|)}{\frac{T}{2}\|\omega(t)\|}}^{\operatorname{sinc}(\frac{T}{2}\|\omega(t)\|)}S\big(\omega(t)\big)\Big)q(t)$$
$$\approx \Big(I_4 + \tfrac{T}{2}S\big(\omega(t)\big)\Big)q(t).$$

The approximation coincides with Euler forward sampling approximation, and has to be used in more complex models where, *e.g.*, ω is part of the state vector.

Double Integrated Quaternion

$$\begin{pmatrix} \dot{q}(t) \\ \dot{\omega}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}S(\omega(t))q(t) \\ w(t) \end{pmatrix}.$$

There is no known closed form discretized model. However, the approximate form can be discretized using the chain rule to

$$\begin{pmatrix} q(t+T) \\ \omega(t+T) \end{pmatrix} \approx \underbrace{\begin{pmatrix} I_4\frac{T}{2}S\big(\omega(t)\big) & \frac{T}{2}\bar{S}\big(q(t)\big) \\ 0_{3\times4} & I_3 \end{pmatrix}}_{F(t)} \begin{pmatrix} q(t) \\ \omega(t) \end{pmatrix}$$
$$+ \underbrace{\begin{pmatrix} \frac{T^3}{4}S\big(\omega(t)\big)I_4 \\ TI_3 \end{pmatrix}}_{G(t)} v(t).$$

Rigid Body Kinematics

A "multi-purpose" model for all kind of navigation problems in 3D (22 states)

$$\begin{pmatrix} \dot{p} \\ \dot{v} \\ \dot{a} \\ \dot{q} \\ \dot{\omega} \\ \dot{b}^{\text{acc}} \\ \dot{b}^{\text{gyro}} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}S(\omega) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ v \\ a \\ q \\ \omega \\ b^{\text{acc}} \\ b^{\text{gyro}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^a \\ v^\omega \\ v^{\text{acc}} \\ v^{\text{gyro}} \end{pmatrix}.$$

Bias states for the accelerometer and gyroscope have been added as well.

Sensor Model for Kinematic Model

Inertial sensors (gyroscope, accelerometer, magnetometer) are used as sensors.

$$\begin{aligned} y_t^{\text{acc}} &= R(q_t)(a_t - \mathbf{g}) + b_t^{\text{acc}} + e_t^{\text{acc}}, & e_t^{\text{acc}} &\sim \mathcal{N}(0, R_t^{\text{acc}}), \\ y_t^{\text{mag}} &= R(q_t)\mathbf{m} + b_t^{\text{mag}} + e_t^{\text{mag}}, & e_t^{\text{mag}} &\sim \mathcal{N}(0, R_t^{\text{mag}}), \\ y_t^{\text{gyro}} &= \omega_t + b_t^{\text{gyro}} + e_t^{\text{gyro}}, & e_t^{\text{gyro}} &\sim \mathcal{N}(0, R_t^{\text{gyro}}). \end{aligned}$$

Bias observable, but special calibration routines are recommended:

Stand-still detection: When $\|y_t^{\text{acc}}\| \approx \mathbf{g}$ and/or $\|y_t^{\text{gyro}}\| \approx 0$, the gyro and acc bias is readily read off. Can decrease drift in dead-reckoning from cubic to linear.

Ellipse fitting: When "waving the sensor" over short time intervals, the gyro can be integrated to give accurate orientation, and acc and magnetometer measurements can be transformed to a sphere from an ellipse.

Tracking Models

- Navigation models have access to inertial information, tracking models have not.
- Orientation mainly the direction of the velocity vector.
- Course (yaw rate) critical parameter.
- Less differences between the 2D and 3D cases.

Coordinated Turns in 2D World Coordinates

Cartesian velocity	Polar velocity
$\dot{X} = v^X$ $\dot{Y} = v^Y$ $\dot{v}^X = -\omega v^Y$ $\dot{v}^Y = \omega v^X$ $\dot{\omega} = 0$	$\dot{X} = v\cos(h)$ $\dot{Y} = v\sin(h)$ $\dot{v} = 0$ $\dot{h} = \omega$ $\dot{\omega} = 0$
$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega & -v^Y \\ 0 & 0 & \omega & 0 & v^X \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$A = \begin{pmatrix} 0 & 0 & \cos(h) & -v\sin(h) & 0 \\ 0 & 0 & \sin(h) & v\cos(h) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$X_{t+T} = X + \frac{v^X}{\omega}\sin(\omega T) - \frac{v^Y}{\omega}(1-\cos(\omega T))$ $Y_{t+T} = Y + \frac{v^X}{\omega}(1-\cos(\omega T)) + \frac{v^Y}{\omega}\sin(\omega T)$ $v^X_{t+T} = v^X\cos(\omega T) - v^Y\sin(\omega T)$ $v^Y_{t+T} = v^X\sin(\omega T) + v^Y\cos(\omega T)$ $\omega_{t+T} = \omega$	$X_{t+T} = X + \frac{2v}{\omega}\sin(\frac{\omega T}{2})\cos(h + \frac{\omega T}{2})$ $Y_{t+T} = Y - \frac{2v}{\omega}\sin(\frac{\omega T}{2})\sin(h + \frac{\omega T}{2})$ $v_{t+T} = v$ $h_{t+T} = h + \omega T$ $\omega_{t+T} = \omega$

Summary

Summary Lecture 5

- Standard models in global coordinates:
  - Translation $p_t^{(m)} = w_t^p$.
  - 2D orientation for heading $h_t^{(m)} = w_t^h$.
  - Coordinated turn model

$$\dot{X} = v^X \qquad \dot{Y} = v^Y$$
$$\dot{v}^X = -\omega v^Y \qquad \dot{v}^Y = \omega v^X$$
$$\dot{\omega} = 0.$$

- Standard models in local coordinates (x, y, ψ):
  - Odometry and dead reckoning for (x, y, ψ)

$$X_t = X_0 + \int_0^t v_\tau^x \cos(\psi_\tau)\, d\tau \qquad Y_t = Y_0 + \int_0^t v_\tau^x \sin(\psi_\tau)\, d\tau$$
$$\psi_t = \psi_0 + \int_0^t \dot{\psi}_\tau\, d\tau.$$

  - Force models for $(\dot{\psi}, a_y, a_x, \dots)$.
  - 3D orientation $\dot{q} = \frac{1}{2}S(\omega)q = \frac{1}{2}\bar{S}(q)\omega$.